Journal of Applied Mechanics and Technical Physics, Vol. 38, No. 2, 1997

APPLICATION OF THE METHOD OF BOUNDARY ELEMENTS AND PARAMETRIC POLYNOMIALS IN AIRFOIL OPTIMIZATION PROBLEMS

S. M. Aul'chenko and A. F. Latypov

UDC 533.6.011

In the present paper, we describe a method for solving design problems of optimum two-dimensional configurations. The method includes a modified variant of the method of boundary elements for solving external-flow problems. Examples of application of this method in designing subsonic airfoils with given characteristics are given.

An important problem of applied sub- and transonic aerodynamics is the design of airfoils with desired aerodynamic characteristics. This necessitates the creation of optimal airfoils with the extreme value of some specific parameter (for example, lift force, lift-to-drag ratio, critical Mach number, pitching moment, etc.). Correctness of the problem formulation determining the existence of a sensible solution and the success in practical design depend on a correct account of constraints. The constraints can be of a different nature: aerodynamic (on the lift force and the pitching moment), gasdynamic (on the nonseparated and subsonic flow regimes), geometric (on the airfoil area, the curvature, and the thickness), and algorithmic (for example, on the smoothness of the functions that specify the airfoil contour). In [1, 2], Aul'chenko and Elizarov et al. used direct methods and the methods for solving inverse boundary-value problems. The authors of the present paper proposed and tested some new approaches to solution of these problems, which were reported in [3–6]. In the present paper, we describe a technique for designing plane optimal configurations, based on the method of boundary elements for calculation of subsonic two-dimensional perfect gas flows. The proposed technique is a continuation of the method of [7] and employs the linear, rather than constant, density distribution of the sources over the boundary elements, thereby increasing the accuracy of calculations with no increase in the number of these elements and in the dimension of the matrix to be inverted. To describe the geometry of the desired boundary which is the airfoil contour to be designed, we propose a technique which is based on the use of fourth-order parametric polynomials with a special choice of conjugate points and which is free of possible oscillations upon variation of the governing parameters and covers a wide class of configurations. Optimization is performed by the gradientless method of search with adaption and with the use of a random element intended for minimization of functions of numerous variables if there are functional constraints in the form of equalities and inequalities [8].

Algorithm for Solving the Flow Problem. Having applied the relations

$$\operatorname{div}\mathbf{V} = \partial V/\partial s + V\partial\theta/\partial n, \quad \partial V/\partial s = V/(\mathrm{M}^2 - 1)\partial\theta/\partial n, \quad \partial V/\partial n = V\partial\theta/\partial s, \tag{1}$$

we can write the differential equations of an inviscid vortex-free gas flow in the form

$$\nabla\mathbf{V} = \mathrm{M}^2\partial V/\partial s, \tag{2}$$

where s is the direction along the tangent to the streamline, n is the normal to it, θ is the angle of inclination of the velocity vector, and M is the local Mach number.

Institute of Theoretical and Applied Mechanics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 38, No. 2, pp. 73–79, March–April, 1997. Original article submitted December 4, 1995; revision submitted December 21, 1995.

After introducing the disturbance potential φ such that $\mathbf{V} = \nabla\Phi + \nabla\varphi$, with allowance for (1) and (2), we obtain

$$\Delta\varphi = \mathrm{M}^2 \partial V/\partial s = Q(\mathrm{M}, V) \tag{3}$$

($\mathbf{V}_\infty = \nabla\Phi$ is the free-stream velocity). The no-slip condition at the airfoil boundary imposes the following requirement on φ:

$$\varphi'_n = -\mathbf{V}_\infty \mathbf{n}. \tag{4}$$

A numerical realization of the Kutta–Joukowski relation for an airfoil with zero angle of the trailing edge is the requirement of equal tangential velocities at the points with radius-vectors $\boldsymbol{\xi}_h$ and $\boldsymbol{\xi}_d$ at the upper and lower airfoil surfaces, respectively, if the condition

$$|\boldsymbol{\xi}_h - \boldsymbol{\xi}_e| = |\boldsymbol{\xi}_d - \boldsymbol{\xi}_e| \sim \varepsilon$$

is satisfied ($\boldsymbol{\xi}_e$ corresponds to the end point of the airfoil and $\varepsilon \ll 1$). The origin of the Cartesian coordinate system $(\mathbf{x}, \mathbf{y})$ is the extreme left point of the airfoil, and the $\mathbf{x}$ axis is directed along its chord.

Using Green's theorem, we can write the potential as

$$\varphi(z) = \operatorname{Re}\Big(\int\limits_C (q + i\,\omega) f(z - z_C)\, dl + \int\limits_D Q f(z - z_C)\, ds\Big) \tag{5}$$

(D is the domain external to the contour C, $f(z) = \ln z/2\pi$, q and ω are the densities of the source and vorticity distributions on the contour, and Q is the source density in D).

On the basis of (5), we can obtain relations for the no-slip conditions at the point $\boldsymbol{\xi}_i$ and also for the Kutta–Joukowski conditions (4):

$$\int\limits_C q(\boldsymbol{\xi})(\partial \ln r(\boldsymbol{\xi}_i, \boldsymbol{\xi})/\partial n_i\, d\xi_x - \partial\theta(\boldsymbol{\xi}_i, \boldsymbol{\xi})/\partial n_i\, d\xi_y) - \omega(\partial\theta(\boldsymbol{\xi}_i, \boldsymbol{\xi})/\partial n_i\, d\xi_x$$
$$+ \partial \ln r(\boldsymbol{\xi}_i, \boldsymbol{\xi})/\partial n_i\, d\xi_y) = -2\pi \mathbf{V}_\infty \mathbf{n} - \int\limits_D Q(s) \partial \ln r(\boldsymbol{\xi}_i, \mathrm{s})/\partial n_i\, ds; \tag{6}$$

$$\int\limits_C [q(\boldsymbol{\xi})(\partial \ln r(\boldsymbol{\xi}_h, \boldsymbol{\xi})/\partial\tau_h - \partial \ln r(\boldsymbol{\xi}_d, \boldsymbol{\xi})/\partial\tau_d)\, d\xi_x - (\partial\theta(\boldsymbol{\xi}_h, \boldsymbol{\xi})/\partial\tau_h - \partial\theta(\boldsymbol{\xi}_d, \boldsymbol{\xi})/\partial\tau_d)]\, d\xi_y$$
$$- \omega[(\partial\theta(\boldsymbol{\xi}_h, \boldsymbol{\xi})/\partial\tau_h - \partial\theta(\boldsymbol{\xi}_d, \boldsymbol{\xi})/\partial\tau_d) d\xi_x + (\partial \ln r(\boldsymbol{\xi}_h, \boldsymbol{\xi})/\partial\tau_h - \partial \ln r(\boldsymbol{\xi}_d, \boldsymbol{\xi})/\partial\tau_d)]\, d\xi_y$$
$$= 2\pi \mathbf{V}_\infty(\boldsymbol{\tau}_h - \boldsymbol{\tau}_d) - \int\limits_D Q(s)(\partial \ln r(\boldsymbol{\xi}_h, \boldsymbol{\xi})/\partial\tau_h - \partial \ln r(\boldsymbol{\xi}_d, \boldsymbol{\xi})/\partial\tau_d)\, ds. \tag{7}$$

Here $r(\boldsymbol{\xi}_i, \boldsymbol{\xi}) = \sqrt{(\xi_{ix} - \xi_x)^2 + (\xi_{iy} - \xi_y)^2}$, $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}$ are the radius-vectors of the corresponding points on the contour, s is the radius-vector of a point in this region, $\theta(\boldsymbol{\xi}_i, \boldsymbol{\xi})$ is the angle between the vectors $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}$. $\mathbf{n}_i$ is the normal at the point $\boldsymbol{\xi}_i$, and $\boldsymbol{\tau}_h$ and $\boldsymbol{\tau}_d$ are the vectors tangent to the contour at the points with radius-vectors $\boldsymbol{\xi}_h$ and $\boldsymbol{\xi}_d$.

Let us split the contour C and the domain D into N and into K elements, respectively. For discretization, the domain D is covered by a nonuniform grid of type C, and, in computations, we also use its subdomain D_K whose length is equal to five lengths of the chord along the $\mathbf{x}$ axis and four lengths of the chord along the $\mathbf{y}$ axis. The computations show that the chosen position of the external boundary of the domain D_K ensures the satisfaction of the undisturbed-flow condition on it with the same accuracy as the solution of the equations. Let the source density at the boundary element l_i be distributed according to the following linear law:

$$q(\eta) = q_{i-1} f_1 + q_i f_2, \quad f_1 = (1 - \eta)/2, \quad f_2 = (1 + \eta)/2, \quad \eta \in (-1, 1).$$

Relation (6) then takes the form

$$\sum_{j=1}^{N} l_j q_{j-1} \int_{-1}^{1} f_1(\eta)(\partial \ln r_{ij}/\partial n_i \, d\eta_x - \partial\theta_{ij}/\partial n_i \, d\eta_y) + \sum_{j=1}^{N} l_j q_j \int_{-1}^{1} f_2(\eta)(\partial \ln r_{ij}/\partial n_i \, d\eta_x - \partial\theta_{ij}/\partial n_i \, d\eta_y)$$

$$-\omega \sum_{j=1}^{N} l_j \int_{-1}^{1} (\partial\theta_{ij}/\partial n_i \, d\eta_x + \partial \ln r_{ij}/\partial n_i \, d\eta_y) = -4\pi \mathbf{V}_\infty \mathbf{n}_i - 2\sum_{k=1}^{K} Q_k \int_{\Delta S_k} \partial \ln r_{ik}/\partial n_i \, ds.$$

Introducing the notation

$$g_{ij}^1 = l_j \int_{-1}^{1} f_1(\eta)(\partial \ln r_{ij}/\partial n_i \, d\eta_x - \partial\theta_{ij}/\partial n_i \, d\eta_y),$$

$$g_{ij}^2 = l_j \int_{-1}^{1} f_2(\eta)(\partial \ln r_{ij}/\partial n_i \, d\eta_x - \partial\theta_{ij}/\partial n_i d\eta_y),$$

$$G_{ij} = g_{ij+1}^1 + g_{ij}^2 \quad (j = 1, \ldots, N-1), \quad G_{iN} = g_{i1}^1 + g_{iN}^2,$$

$$G_{iN+1} = \sum_{j=1}^{N} l_j \int_{-1}^{1} (\partial\theta_{ij}/\partial n_i \, d\eta_x + \partial \ln r_{ij}/\partial n_i \, d\eta_y),$$

$$H_i = -4\pi \mathbf{V}_\infty \mathbf{n}_i - 2\sum_{k=1}^{K} Q_k \int_{\Delta S_k} \partial \ln r_{ik}/\partial n_i \, ds,$$

we obtain the following system of algebraic equations for N collocation points $\boldsymbol{\xi}_i$ $(i = 1, \ldots, N)$ on the contour:

$$\mathbf{G}\mathbf{q} = \mathbf{H}. \tag{8}$$

The coefficient matrix $\mathbf{G}$ consists of the elements G_{ij} $(i = 1, \ldots, N$ and $j = 1, \ldots, N+1)$; the vector on the right-hand sides is $\mathbf{H} = (H_1, \ldots, H_N)$, and the vector of solution is $\mathbf{q} = (q_1, \ldots, q_N, \omega)$. A standard form of the equation closing system (8) is found from (7) using the following notation:

$$g_{hj}^1 = l_j \int_{-1}^{1} f_1(\partial \ln r_{hj}/\partial\tau_h \, d\eta_x - \partial\theta_{hj}/\partial\tau_h \, d\eta_y), \; g_{hj}^2 = l_j \int_{-1}^{1} f_2(\partial \ln r_{hj}/\partial\tau_h \, d\eta_x - \partial\theta_{hj}/\partial\tau_h \, d\eta_y),$$

$$g_{dj}^1 = l_j \int_{-1}^{1} f_1(\partial \ln r_{dj}/\partial\tau_d \, d\eta_x - \partial\theta_{dj}/\partial\tau_d \, d\eta_y), \quad g_{dj}^2 = l_j \int_{-1}^{1} f_2(\partial \ln r_{dj}/\partial\tau_d \, d\eta_x - \partial\theta_{dj}/\partial\tau_d \, d\eta_y),$$

$$g_{hj}^{12} = g_{hj+1}^1 + g_{hj}^2, \qquad g_{dj}^{12} = g_{dj+1}^1 + g_{dj}^2, \qquad g_{hN}^{12} = g_{h1}^1 + g_{hN}^2, \qquad g_{dN}^{12} = g_{d1}^1 + g_{dN}^2,$$

$$G_{N+1,j} = g_{hj}^{12} - g_{dj}^{12} \quad (j = 1, \ldots, N),$$

$$G_{N+1,N+1} = -\sum_{j=1}^{N} l_j \int_{-1}^{1} (\partial\theta_{hj}/\partial\tau_h - \partial\theta_{dj}/\partial\tau_d)\, d\eta_x + (\partial \ln r_{hj}/\partial\tau_h - \partial \ln r_{dj}/\partial\tau_d)\, d\eta_y,$$

$$H_{N+1} = 4\pi \mathbf{V}_\infty(\boldsymbol{\tau}_h - \boldsymbol{\tau}_d) - 2\sum_{k=1}^{K} Q_k \int_{\Delta S_k} (\partial \ln r_{hk}/\partial\tau_h - \partial \ln r_{dk}/\partial\tau_d)\, ds.$$

The resulting system is solved by the method of simple iterations over the nonlinear right-hand side. The initial value of Q in the domain is assumed to be equal to zero, which corresponds to the incompressible flow. The following procedure for calculating Q_k in the cells ΔS_k of the domain in each iteration is used.

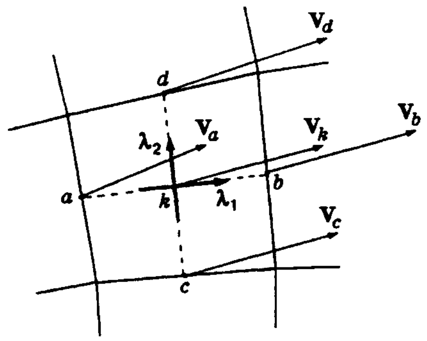

Fig. 1

Differentiation of (5) with respect to $\mathbf{x}$ and $\mathbf{y}$ at the boundary points and at the cell center ΔS_k (Fig. 1) yields the velocity values $\mathbf{V}_a$, $\mathbf{V}_b$, $\mathbf{V}_c$, $\mathbf{V}_d$, and $\mathbf{V}_k$. In the finite-difference approximation, we have

$$\partial V_k/\partial s(\mathbf{s}\boldsymbol{\lambda}_1) + \partial V_k/\partial n(\mathbf{n}\boldsymbol{\lambda}_1) = \partial V_k/\partial \lambda_1 = (V_b - V_a)/ba,$$

$$\partial V_k/\partial s(\mathbf{s}\boldsymbol{\lambda}_2) + \partial V_k/\partial n(\mathbf{n}\boldsymbol{\lambda}_2) = \partial V_k/\partial \lambda_2 = (V_d - V_c)/dc,$$

whence

$$\partial V_k/\partial s = (\partial V_k/\partial \lambda_1(\mathbf{n}\boldsymbol{\lambda}_2) - \partial V_k/\partial \lambda_2(\mathbf{n}\boldsymbol{\lambda}_1))/((\mathbf{s}\boldsymbol{\lambda}_1)(\mathbf{n}\boldsymbol{\lambda}_1) - (\mathbf{s}\boldsymbol{\lambda}_2)(\mathbf{n}\boldsymbol{\lambda}_1)).$$

Then

$$Q_k = [(V_k/V_\infty)^2 \mathrm{M}_\infty^2/(1 - ((\gamma - 1)/2)\mathrm{M}_\infty^2(1 - (V_k/V_\infty)^2))]\partial V_k/\partial s.$$

Representation of the Geometry. In design problems of optimal configurations, it is always necessary to describe the desired boundary. In problems of hydrodynamics and subsonic aerogasdynamics, the functions should usually have at least first-order smoothness. In the course of their determination, depending on the method of representation, one needs to vary the governing parameters: either nodes' coordinates or polynomial coefficients, etc. Generally speaking, this can induce "parasitic" oscillations, which deteriorates the correctness of optimization problems. There are various methods [9] for designing curves and surfaces, among which we note the Fergusson–Bernstein–Bezier method of polynomial curves. However, a large number of governing parameters are required for generation of a desired boundary from numerical solution of optimization problems. This leads, in particular, to high-order Bezier curves, which weakens substantially the link between the characteristic broken line and the line obtained. A method based on the use of parametric polynomials with a special choice of the conjugation points of interpolation intervals which is exempt from the above-mentioned disadvantages is suggested. This method is described in detail in [10].

The airfoil contour is constructed as follows. For its upper part, the values of $r_h, x_1^h, \ldots, x_N^h, y_1^h, \ldots, y_N^h$, are set. Here r_h is the radius of curvature of the contour head, x_i^h and y_i^h are the abscissas and ordinates of the frame nodes (the center of the circle lies on the $\mathbf{x}$ axis), and $x_N^h = 1$ and $y_N^h = 0$. The coordinates of the point of conjugation of the arc at the airfoil head with the first section of the frame are calculated with the requirements of first-order smoothness taken into account. Subsequent construction of the curve has a local character: on $[x_{i-1}^h, x_i^h]$ and $[x_i^h, x_{i+1}^h]$, the nodes z_{i-1}^h and z_i^h are chosen from the condition of a constant-sign second derivative at which the conjugation of first-order smoothness with the neighboring segments is ensured in these nodes, and a curve specified by a fourth-order parametric polynomial is constructed on the section $[z_{i-1}^h, z_i^h]$. The above algorithm for constructing curves inscribed in the frame is repeated from the first to the end point. The varied parameters for optimization are r_h, x_i^h, and y_i^h, and the quantities η_i^h which determine z_i^h inside the allowable interval. The lower section of the contour is constructed in a similar way.

TABLE 1

Ordinal number	M_∞	K_a	C_l	C_d	C_m	d_{max}, %	S_C
1	0.5	82.1	0.586	0.007	−0.245	11.9	0.074
2	0.7	62.2	0.402	0.006	−0.191	11.2	0.072
3	0.5	15.0	1.198	0.079	−0.510	11.3	0.066
4	0.5	29.4	0.600	0.020	−0.019	11.6	0.068

Formulation of Optimization Problems. Let us design an airfoil that satisfies the following aerodynamic

$$M_\infty = \text{const}, \quad M < 1, \quad C_l > C_l^0, \quad f(s) > f_0, \quad \alpha_0 < \alpha < \alpha_1,$$

and geometric

$$d_{max} < d_1, \quad S_0 < S_C, \quad b = \text{const}, \quad \text{contour} \quad C \in C^1(0,1)$$

constraints and provides a minimum (maximum) to a certain objective function. Here M_∞ is the free-stream Mach number, $f(s)$ is the form factor, s is the length of the arc along the contour, f_0 is an empirical constant correlated with the chosen flow-without-separation criterion, α is the angle of attack relative to the chord. d_{max} is the maximum airfoil thickness in percent, S_C is the airfoil area, and b is the length of the airfoil chord. As objective functionals, we chose the maximum lift-to-drag ratio $F_1 = K_a$, the maximum lift coefficient $F_2 = C_l$, and the minimum pitching moment $F_3 = C_m$ ($K_a = C_l/C_d$, where C_d is the drag coefficient).

One can substantially facilitate the process of airfoil design taking into account the fact that the flow past wings occurs mainly at high (of the order of 10^5–10^6) Reynolds numbers. In such a flow regime, the viscosity exerts some effect only in a fairly thin layer, and, hence, should be taken into account within the framework of the boundary-layer model. The C_d value can be approximately calculated using the known Squire–Young's formula, and the value of f_0 can be taken from the Kochin–Loitsyanskii flow-without-separation criterion.

Calculation Results. Figure 2 shows the designed airfoil possessing a maximum lift-to-drag ratio for the following values in constraints: $M_\infty = 0.5$, $C_l^0 = 0$, $f_0 = -2.5$, $\alpha_0 = -5°$, $\alpha_1 = 20°$, $d_{max} < 12\%$. $S_0 = 0.06$, and $b = 1$. As the initial airfoil, we used an arbitrary symmetric contour in flow at zero angle of attack. This figure also shows the distribution of the pressure coefficient C_p over the upper and lower contours. The integral and geometric characteristics of the optimal airfoil obtained are given in Table 1 (row No. 1).

Figure 3 shows the designed airfoil possessing a maximum lift-to-drag ratio with the same values of the constraints except for $M_\infty = 0.7$. The optimal contour from the previous problem was taken as the original countor. The integral and geometric characteristics of the modified airfoil are listed in Table 1 (row No. 2). In the pressure-coefficient distribution on the upper contour, one can see that the critical value of $C_p^* = -0.78$ is reached. The airfoil itself has the negative curvature −0.372%, which is typical of transonic airfoils.

The designed airfoil with a maximum lift force for $M_\infty = 0.5$ and under the above-listed constraints is shown in Fig. 4. The optimal contour from the first problem was again taken as the original one. The airfoil characteristics obtained are given in Table 1 (row No. 3). Compared with the original airfoil (row No. 1), the coefficient C_l is increased by a factor of two, and the displacement of the tongue of concavity to the extreme positions typical of airfoils with high C_l is observed. For the airfoil in Fig. 4, the abscissa of its maximum curvature is $x_f = 0.14$.

Figure 5 shows the designed airfoil with a minimum value of the pitching-moment coefficient C_m with the lift-force coefficient bounded from below $C_l > 0.6$. The remaining constraints are the same. The previous optimal contour was taken as the original one. The characteristics are listed in Table 1 (row No. 4). Clearly. the considerable decrease in C_m is caused by a decrease in C_l (its value is outside the boundary) and also because the center of pressure is shifted toward the front part of the airfoil: the abscissa of the center of pressure is 0.329 for the contour obtained and 0.424 for the original one. The number of the boundary elements N on

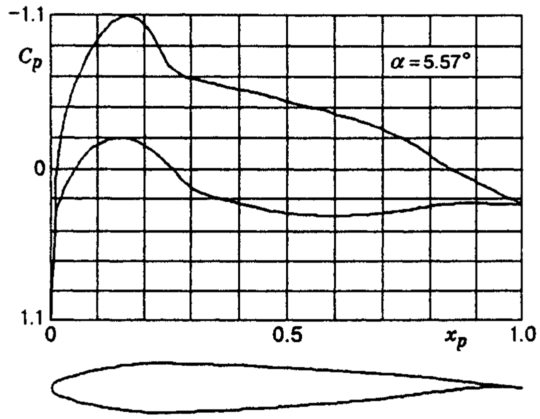

Fig. 2

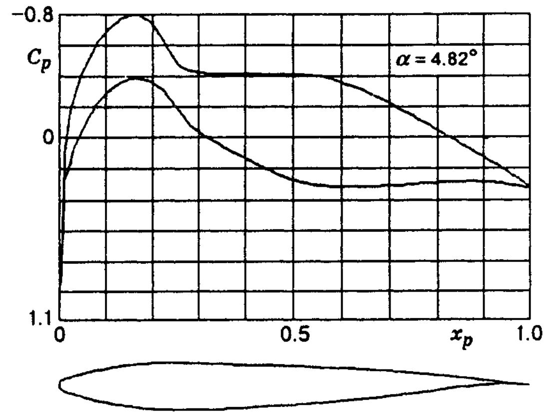

Fig. 3

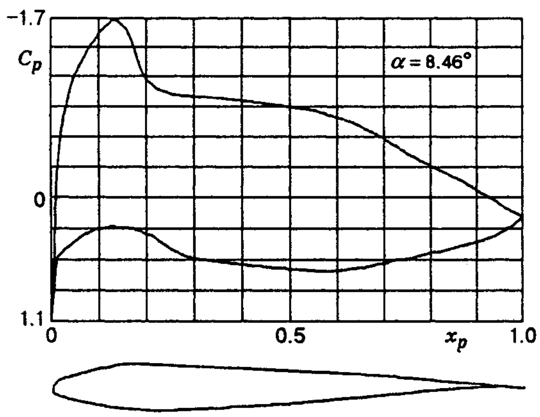

Fig. 4

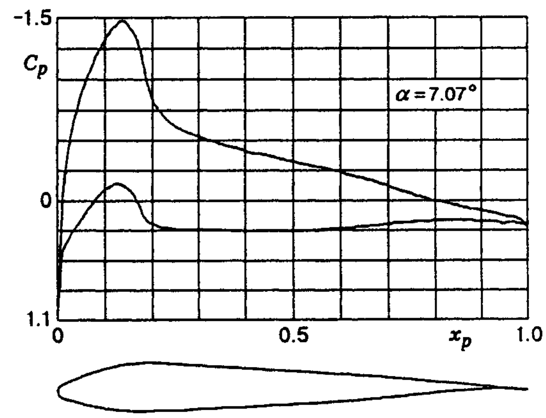

Fig. 5

the contour C was 70, and the number of the cells K in the subdomain D_K was 1360.

In conclusion, we note the following. When direct methods are used to solve optimization problems, the decisive condition of the efficiency of algorithms for flow calculations is the uniform accuracy for the entire set of configurations to be calculated in the course of the search. The proposed algorithm for solving the direct problem satisfies this requirement, since the structure of the inverted matrix **G** is not dependent on the discretization of the domain, and, hence, the possibility of losing the accuracy of calculation of functionals with arbitrary variation in the free boundary is eliminated. This advantage of the algorithm over finite-difference and finite-element methods makes it possible to obtain solutions for finite variations of the geometric parameters of the original contour, which is illustrated by the above calculations. This circumstance is of importance, since owing to the nonlinear dependence of the functionals on the airfoil geometry and also to the presence of constraints, the solutions obtained are the local extrema of the problem.

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